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# Finite temperature and density corrections to quantum electrodynamics†

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**Abstract.** The effective Lagrangian and the electric current of quantum electrodynamics (QED) at weak electromagnetic (EM) field and low temperature is investigated. With the help of the effective Lagrangian the permittivity and permeability are obtained. The effective potential at finite temperature and density induced by the classical Coulomb potential is also obtained. It is shown that the heat corrections modify the Coulomb potential to such an extent that the effective potential adopts the Yukawa form. Some remarks are made about the electron $g$ factor at finite temperature and density.

## 1. Introduction

Interest in the finite temperature and density corrections to the models of quantum field theory has been initiated by the suggestion of Kirzhnits and Linde (1972) that a spontaneously broken symmetry, in a relativistic field theory coupled to a finite-temperature heat bath, could be restored above some critical temperature; and by the qualitative calculations performed in the one-loop approximation by Weinberg (1974) and Dolan and Jackiw (1974). It has also been recognised that a similar phenomenon occurs at high density of leptons and baryons (Harrington and Yildiz 1974). Due to the enormous mass of the Higgs boson, the aforementioned restoration of the spontaneously broken symmetry occurs at temperatures and densities of the order of $10^{15}$ K and $10^{47}$ cm$^{-3}$, respectively. Such conditions can only be found in the early stages of the Universe; therefore, these considerations have only cosmological applications.

On the other hand, taking into account the suggestion of Ritus (1978) that the effective Lagrangian of QED in strong EM field can be used for the investigation of the fundamental questions of QED (like the finiteness of the renormalisation constants), I have studied the finite temperature and density corrections to QED, proving that the effective Lagrangian at weak EM field and high temperature determines the Johnson–Baker–Willey function $F^{[1]}$§ (Kamiński 1981a, 1982b), and conjecturing this statement in the case of weak EM field and high chemical potential with respect to the temperature (Kamiński 1982b).

The investigation of QED at finite temperature and density is important, since QED in a vacuum is so far the only known example of the relativistic field theory which

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§ This function plays the fundamental role in the Johnson–Baker–Willey model of QED (e.g. Adler 1972).

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gives excellent agreement with experiment (Lautrup *et al* 1972). This is the reason why temperature corrections to the measurable quantities are studied.

This paper is the continuation of a previous investigation of QED at low temperatures. It was shown (Kamiński 1981b) that at more realistic temperatures ($T \ll m \approx 5.8\times 10^9$ K), temperature corrections to QED processes are extremely small and are proportional to a power of $\exp(-m/T)$. In this paper I show that density corrections temper such a decreasing of the temperature-dependent part of the quantities under consideration. The finite-temperature corrections to the effective Lagrangian have been also investigated by Bauhoff and Dittrich (1981), and Dittrich (1979a, b).

The reader who is not familiar with the finite temperature and density methods in QED is referred to Bechler (1981) and references therein.

## 2. Effective Lagrangian

The effective Lagrangian $\mathscr{L}_{\text{eff}}$ at finite temperature and density, and constant EM field†, is defined by the relation:

$$\exp(\beta V \mathscr{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu)) = \mathrm{Sp}\,\exp[+\beta(\mathscr{L}_{\text{QED}} + \mu N)] \tag{1}$$

where $\mathscr{L}_{\text{QED}}$ and $N$ are the QED Lagrangian and the fermion number operator respectively. $\mathscr{L}_{\text{eff}}$ is a function of the electric field $\boldsymbol{E}$, magnetic induction $\boldsymbol{B}$, temperature $T = \beta^{-1}$ and chemical potential $\mu$ ($V$ is the volume of the system under consideration). Sp denotes the sum over all possible quantum states of the EM and electron fields. Furthermore, we will work within the approximation of neglecting radiative corrections. This means that in (1) the summation is over the quantum states of the electron field. Within this approximation it is straightforward to show that (Kamiński 1981c):

$$\begin{aligned}\mathscr{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu) &= (\boldsymbol{E}^2 - \boldsymbol{B}^2)/2 - T\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\int_m^{\infty}\mathrm{d}m\\ &\quad\times \mathrm{Tr}\,\tilde{K}_{\mathrm{F}}[\mu + 2\pi \mathrm{i} T(n+\tfrac{1}{2}), \boldsymbol{p}|\boldsymbol{E}, \boldsymbol{B}] + \mathrm{CT}\end{aligned} \tag{2}$$

where the finite temperature and density methods have been used (Bechler 1981). The symbol CT denotes the contact terms, defined in such a way that the effective Lagrangian vanishes at vanishing $\boldsymbol{E}$, $\boldsymbol{B}$ and $T$; and Tr denotes the trace of the Dirac matrices. Moreover, $\tilde{K}_{\mathrm{F}}[p_0, p|\boldsymbol{E}, \boldsymbol{B}]$ is the Fourier transform of the Feynman propagator of the electron in the constant EM field (see appendix). With the help of (2) one can arrive at:

$$\mathscr{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu) = (\boldsymbol{E}^2 - \boldsymbol{B}^2)/2 + \mathscr{L}_{\text{HE}}^{(1)}(\boldsymbol{E}, \boldsymbol{B}) + \Delta\mathscr{L}^{(1)}(\boldsymbol{E}, \boldsymbol{B}, T, \mu) \tag{3}$$

where the renormalisation of the electron charge has been performed. Within the approximation of neglecting radiative corrections, we need not perform the renormalisation of the electron mass. $\mathscr{L}_{\text{HE}}^{(1)}(\boldsymbol{E}, \boldsymbol{B})$ is the Heisenberg–Euler correction to the

† It is sufficient if the EM field is a little changing function,

$$|f_{\mu\nu}(t+\lambda_c, \boldsymbol{r}+\lambda_c \boldsymbol{r}/r) - f_{\mu\nu}(t, \boldsymbol{r})| \ll |f_{\mu\nu}(t, \boldsymbol{r})|$$

where $\lambda_c$ is the electron Compton wavelength and $f_{\mu\nu}$ is the electromagnetic field tensor.

free Lagrangian and $\Delta\mathscr{L}^{(1)}(\boldsymbol{E}, \boldsymbol{B}, T, \mu)$ is the finite temperature and density correction. The Heisenberg–Euler correction has been extensively investigated both from the fundamental (Dittrich 1976, 1977, Ritus 1978, Kamiński 1981c, 1982a, Hagiwara 1981) and the applied (Adler 1971, Białynicka-Birula and Białynicki-Birula 1970, Mielniczuk 1982) point of view. Taking advantage of the Poisson summation formula (Schwartz 1966),

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int \mathrm{d}x\, f(x) \exp(2\pi n \mathrm{i} x)$$

from which follows the identity

$$\sum_{n=-\infty}^{\infty} \exp\{\mathrm{i}s[\mu + 2\pi \mathrm{i} T(n+\tfrac{1}{2})]^2\} = \beta(4\pi \mathrm{i}s)^{-1/2}\theta_4(\mathrm{i}\beta\mu/2, \exp[-\pi\beta^2(4\pi\mathrm{i}s)^{-1}]),$$

where $\theta_4$ is the elliptic theta function (Abramowitz and Stegun 1965), one finds the following form of the finite temperature and density correction $\Delta\mathscr{L}^{(1)}$

$$\Delta\mathscr{L}^{(1)} = 2T^4 \int_0^{\infty} \mathrm{d}s\, s^{-1}\mathscr{E}\mathscr{B} \cot(\mathscr{E}s) \coth(\mathscr{B}s) \exp(-\pi s \nu^2)$$
$$\times (1 - \theta_4\{\mathrm{i}\pi\rho, \exp[-\pi\mathscr{E} \cot(\mathscr{E}s)]\}) \tag{4}$$

where $\mathscr{E} = eE\beta^2/4\pi$, $\mathscr{B} = eB\beta^2/4\pi$, $\nu = m\beta/2\pi$, $\rho = \mu\beta/2\pi$, and $E$ and $B$ are the electric and magnetic fields in the Lorentz frame in which they are parallel to each other.

At zero temperature, derivatives of the effective Lagrangian with respect to $\boldsymbol{E}$ and $\boldsymbol{B}$ give the electric displacement $\boldsymbol{D}$ and the magnetic field intensity $\boldsymbol{H}$, respectively. These properties are assumed to be valid at finite temperature and density, i.e., by definition,

$$D_i = (\partial/\partial E_i)\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu), \tag{5}$$

$$H_i = (\partial/\partial B_i)\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu). \tag{6}$$

Moreover, it follows from (1) that the number density (derived with the help of the electric current in the appendix) is given by

$$n = (\partial/\partial\mu)\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu). \tag{7}$$

## 3. Effective Lagrangian at low temperature and weak electric field

This section is concerned with the low-temperature ($T \ll 5.8 \times 10^9$ K) and weak electric field ($E \ll Tm/e$) corrections to the effective Lagrangian. With the help of (4) one finds

$$\Delta\mathscr{L}^{(1)} = -4T^4 \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (-1)^n \cosh(2\pi n\rho) W_{kn}(\mathscr{E}, \mathscr{B})$$
$$\times \int_0^{\infty} \mathrm{d}s\, s^{k-3} \exp(-\pi s\tilde{\nu}^2 - \pi n^2/s) \tag{8}$$

where

$$\tilde{\nu}^2 = \nu^2 - \mathscr{E}^2/3 \tag{9}$$

and

$$\sum_{k=0}^{\infty} s^k W_{kn}(\mathscr{E}, \mathscr{B})$$
$$= s^2 \mathscr{E}\mathscr{B} \cot(\mathscr{E}s) \coth(\mathscr{B}s) \exp\{-\pi n^2[(\mathscr{E}s)\cot(\mathscr{E}s) - 1 + \mathscr{E}^2 s^2/3]/s\}.$$

Taking advantage of the integral representation of the modified Bessel functions (Abramowitz and Stegun 1965)

$$\int_0^{\infty} \mathrm{d}x\, x^{\nu-1} \exp[-(\beta/x) - \gamma x] = 2(\beta/\gamma)^{\nu/2} K_\nu[2(\beta\gamma)^{1/2}],$$

their asymptotic expansion

$$K_\nu(z) \propto (\pi/2z)^{1/2}\, \mathrm{e}^{-z}$$

and performing the summation over the index $k$, we obtain

$$\Delta\mathscr{L}^{(1)} = -4T^4 \tilde{\nu}^{3/2} \sum_{n=1}^{\infty} (-1)^n n^{-5/2} \cosh(2\pi n\rho) \exp(-2\pi n\tilde{\nu})$$
$$\times (\mathscr{E}n\tilde{\nu}^{-1})(\mathscr{B}n\tilde{\nu}^{-1}) \cot(\mathscr{E}n\tilde{\nu}^{-1}) \coth(\mathscr{B}n\tilde{\nu}^{-1})$$
$$\times \exp\{-\pi n\tilde{\nu}[\mathscr{E}n\tilde{\nu}^{-1}\cot(\mathscr{E}n\tilde{\nu}^{-1}) - 1 + \mathscr{E}^2 n^2 \tilde{\nu}^{-2}/3]\}. \tag{10}$$

At vanishing electric field and small $\rho$ the leading term ($n = 1$) of (10) gives

$$\Delta\mathscr{L}^{(1)} = 2T^4 (m/2\pi T)^{3/2} (eB/2mT) \coth(eB/2mT)(z_- + z_+), \tag{11}$$

where

$$z_\pm = \exp[-(m \pm \mu)/T], \tag{12}$$

i.e., the result I have obtained at $\mu = 0$ with the help of the steepest descent method (Kamiński 1981b). Since

$$F(-z, s) = -\frac{z}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t \frac{t^{s-1}\,\mathrm{e}^{-t}}{1 + z\,\mathrm{e}^{-t}} = \sum_{n=1}^{\infty} (-1)^n n^{-s} z^n \tag{13}$$

one finds the following form of the finite temperature and density correction to the effective Lagrangian at weak EM field:

$$\Delta\mathscr{L}^{(1)} = -2T^4 \tilde{\nu}^{3/2} [\tilde{F}(\tfrac{5}{2}) - \tfrac{1}{3}\tilde{\nu}^{-2}(\mathscr{E}^2 - \mathscr{B}^2)\tilde{F}(\tfrac{1}{2})$$
$$- \tfrac{1}{45}\pi\tilde{\nu}^{-3}\mathscr{E}^4 \tilde{F}(-\tfrac{1}{2}) + \tfrac{1}{45}\tilde{\nu}^{-4}(\mathscr{E}^4 - 5\mathscr{E}^2\mathscr{B}^2 + \mathscr{B}^4)\tilde{F}(-\tfrac{3}{2})] \tag{14}$$

where $\tilde{F}(s) = F(-\tilde{z}_+, s) + F(-\tilde{z}_-, s)$ and $\tilde{z}_\pm = \exp(-2\pi(\tilde{\nu} \pm \rho))$.

We now consider the following cases.

(a) $\rho \sim 1$

In this case, taking only the leading term ($n = 1$) of the sum (13), one obtains the result that could be derived with the help of the steepest descent method, i.e. the finite temperature and density corrections to the effective Lagrangian are extremely small and can be neglected.

(b) $|\rho - \tilde{\nu}| \ll 1$

Using the relation

$$F(-1 - \delta z, s) = (2^{1-s} - 1)\zeta(s) + (2^{2-s} - 1)\zeta(s-1)\delta z,$$

where $|\delta z| \ll 1$ and $\zeta(s)$ is the Riemann zeta function, one obtains

$$\begin{aligned}\Delta\mathscr{L}^{(1)} = 2T^4\tilde{\nu}^{3/2}\{&(1-2^{-3/2})\zeta(\tfrac{5}{2}) + 2\pi(\rho-\tilde{\nu})(1-2^{-1/2})\zeta(\tfrac{3}{2}) - \tfrac{1}{3}\tilde{\nu}^{-2}(\mathscr{E}^2-\mathscr{B}^2)\\ &\times[(1-2^{1/2})\zeta(\tfrac{1}{2}) + 2\pi(\rho-\tilde{\nu})(1-2^{3/2})\zeta(-\tfrac{1}{2})]\\ &-\tfrac{1}{45}\pi\tilde{\nu}^{-3}\mathscr{E}^4[(1-2^{3/2})\zeta(-\tfrac{1}{2}) + 2\pi(\rho-\tilde{\nu})(1-2^{5/2})\zeta(-\tfrac{3}{2})]\\ &+\tfrac{1}{45}\tilde{\nu}^{-4}(\mathscr{E}^4 - 5\mathscr{E}^2\mathscr{B}^2 + \mathscr{B}^4)[(1-2^{5/2})\zeta(-\tfrac{3}{2}) + 2\pi(\rho-\tilde{\nu})(1-2^{7/2})\zeta(-\tfrac{5}{2})]\}.\end{aligned} \tag{15}$$

The number density at the vanishing EM field is equal to

$$n = 2(1-2^{-1/2})\zeta(\tfrac{3}{2})(mT/2\pi)^{3/2}.$$

This means that in a volume of the order of $10^{-39}$ cm$^3$ there are $(T/m)^{3/2}$ fermions. Such densities can be encountered in the present Universe†.

(c) $\rho - \tilde{\nu} \gg 1$

Using the relations (Bateman 1953)

$$F(z,s) + e^{i\pi s}F(z^{-1},s) = (2\pi)^s\Gamma(s)^{-1}\, e^{i\pi s/2}\zeta(1-s, (\ln z)/2\pi i)$$

$$\zeta(s,v) \propto \Gamma(s)^{-1}[v^{1-s}\Gamma(s-1) + O(v^{-s})], \qquad |v| \gg 1,$$

one arrives at

$$\begin{aligned}\Delta\mathscr{L}^{(1)} = 2^{-1/2}\pi^{-2}T^4(m/T)^{3/2}&[1-\tfrac{1}{16}(Em/F_0T)^2]\\ &\times\left[\frac{8}{15}\left(\frac{\mu}{T} - \frac{m}{T} + \frac{1}{24}\frac{E^2}{F_0^2}\frac{m^3}{T^3}\right)^{5/2}\right.\\ &\left.-\frac{1}{6}\frac{E^2-B^2}{F_0^2}\frac{m^2}{T^2}\left(\frac{\mu}{T} - \frac{m}{T} + \frac{1}{24}\frac{E^2}{F_0^2}\frac{m^3}{T^3}\right)^{1/2}\right],\end{aligned} \tag{16}$$

where $F_0 = m^2/e \sim 10^{13}$ Gs. It follows from (16) that at the vanishing EM field in the volume of the order of $10^{-39}$ cm$^3$ there are $(\mu/m)^{3/2}$ fermions, i.e., we encounter densities of the primeval cosmological matter.

Let me summarise the results obtained in this section and write down the effective Lagrangian at weak EM field in the following form:

$$\mathscr{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu) = \Delta_0(T,\mu) + \tfrac{1}{2}\boldsymbol{E}^2[1+\Delta_2^D(T,\mu)] - \tfrac{1}{2}\boldsymbol{B}^2[1+\Delta_2^B(T,\mu)], \tag{17}$$

where

$$\Delta_0(T,\mu) = -2T^4(m/2\pi T)^{3/2}F(\tfrac{5}{2}),$$

$$\Delta_2^D(T,\mu) = F_0^{-2}\,(m/2\pi T)^{3/2}[\tfrac{1}{4}m^2T^2F(\tfrac{5}{2}) + \tfrac{1}{6}m^3TF(\tfrac{3}{2}) + \tfrac{2}{3}m^2T^2F(\tfrac{1}{2})],$$

$$\Delta^B(T,\mu) = -\tfrac{2}{3}F_0^{-2}\,(m/2\pi T)^{3/2}m^2T^2F(\tfrac{1}{2})$$

and $F(s) = F(-z_+, s) + F(-z_-, s)$. It follows from (17) that the permittivity $\varepsilon_D$ and permeability $\mu_B$ are equal to

$$\varepsilon_D = 1 + \Delta_2^D(T,\mu), \qquad \mu_B = [1+\Delta_2^B(T,\mu)]^{-1},$$

i.e., at small temperature and density the refractive index is less than 1. Using (17)

† There are at least two places in the Universe (namely, the neutron star cores and the primeval cosmological matter) where densities several times greater than the nuclear one are predicted by the standard models.

one can also find the energy density of the EM field

$$\mathscr{H}_{\mathrm{EM}}(\boldsymbol{r}, t) = \tfrac{1}{2}\varepsilon_D^{-1}\boldsymbol{D}^2(\boldsymbol{r}, t) + \tfrac{1}{2}\mu_B^{-1}\boldsymbol{B}^2(\boldsymbol{r}, t) \tag{18}$$

where the dependence of the EM field on $\boldsymbol{r}$ and $t$ has been written explicitly (within, of course, the approximation under consideration).

## 4. Electric current

It was shown by Schwinger (1951) that at zero temperature the constant electric and/or magnetic fields (parallel to each other) can create an electron–positron pair and the probability of such a creation is equal to

$$P(\boldsymbol{E}, \boldsymbol{B}) = 1 - \exp\left(-2\int \mathrm{d}^4x \ \mathrm{Im}\,\mathscr{L}_{\mathrm{HE}}(\boldsymbol{E}, \boldsymbol{B})\right). \tag{19}$$

This phenomenon may be understood on the semiclassical level. (Since the creation of an electron–positron pair is not a classical phenomenon this description is called the semiclassical one.) To this end let me consider Newton's law with the Lorentz force on the right-hand side. Assume that at $t = 0$ and $\boldsymbol{r} = 0$ an electron–positron pair is created (the coordinate frame is chosen in such a manner that $\boldsymbol{E}\|\boldsymbol{B}\|0_z$) and the initial velocities of fermions are equal to zero. It can be immediately checked that the trajectories of the electron $\xi_\parallel^{(-)}$ and positron $\xi_\parallel^{(+)}$ are described by the equations

$$\xi_\parallel^{(-)}: x_1^{(-)}(t) = x_2^{(-)}(t) = 0, \qquad x_3^{(-)}(t) = -\tfrac{1}{2}(|e|E/m)t^2, \tag{20a}$$

$$\xi_\parallel^{(+)}: x_1^{(+)}(t) = x_2^{(+)}(t) = 0, \qquad x_3^{(+)}(t) = \tfrac{1}{2}(|e|E/m)t^2. \tag{20b}$$

(The EM field is assumed to be very weak, so this problem can be treated non-relativistically.) We see that for $t > 0$ the trajectories never intersect each other, therefore, the electron–positron pairs are created.

In the case of the crossed EM field ($\boldsymbol{E} \perp \boldsymbol{B}, |\boldsymbol{E}| = |\boldsymbol{B}| = B$) the situation is quite different. The trajectories are described by the equations

$$\xi_\perp^{(-)}: x_1^{(-)}(t) = t + \omega^{-1}\sin\omega t, \qquad x_2^{(-)}(t) = -\omega^{-1}(1-\cos\omega t), \qquad x_3^{(-)}(t) = 0, \tag{21a}$$

$$\xi_\perp^{(+)}: x_1^{(+)}(t) = t + \omega^{-1}\sin\omega t, \qquad x_2^{(+)}(t) = \omega^{-1}(1-\cos\omega t), \qquad x_3^{(+)}(t) = 0, \tag{21b}$$

where $\omega = |e|B/m$. So we see that the trajectories $\xi_\perp^{(\pm)}$ intersect each other at the point $t = x_1 = 2\pi\omega^{-1}$, $x_2 = x_3 = 0$, therefore, the annihilation of the electron–positron pair occurs. This means that the crossed EM field does not create the electron–positron pairs. At finite temperatures and densities the situation differs from the zero temperature case. The trajectories $\xi_\perp^{(\pm)}$ are disturbed since the projectiles are scattered by the real particles. (The interaction of the projectile with the virtual pairs does not change its momentum, i.e., the trajectory is undisturbed. This means that the creation of pairs occurs only at non-vanishing $\mu$, as will be seen in the following.) Therefore, there exists a non-vanishing probability that the trajectories do not intersect each other, i.e., a non-vanishing probability of the pair creation. This phenomenon can be clarified from the quantum point of view. To this end we consider the Feynman propagator of the electron in the crossed EM field†,

$$K_F^{\perp}[x, x'|\boldsymbol{E}] = \int \frac{d^4p}{(2\pi)^4} \exp[-ip(x-x')]\left[m + \gamma p - ie(\gamma a)k^\mu \frac{\partial}{\partial p^\mu}\right] \times (m^2 - p^2 - i\varepsilon)^{-1} f_p[kx, kx'|\boldsymbol{E}] \tag{22}$$

where

$$f_p[kx, kx'|\boldsymbol{E}] = \exp\left(-i\int_0^{k(x-x')} d\theta \frac{1}{2pk}[2epa\theta - e^2a^2\theta^2 + ie(\gamma k)(\gamma a)\theta]\right),$$

and $a^0 = 0$, $\boldsymbol{E} = -\omega \boldsymbol{a}$, $\boldsymbol{B} = \boldsymbol{k} \times \boldsymbol{E}/\omega$. Taking advantage of the finite temperature and density methods one finds that the electric current $\boldsymbol{j}(x)$ is a complex quantity which causes the creation of the electron–positron pairs (Białynicki-Birula and Białynicka-Birula 1975). Moreover, as has been previously predicted using semiclassical considerations, the electric current vanishes at $\mu = 0$. The qualitative predictions of this statement are now studied and will be presented in due course.

## 5. Effective potential

The aim of this section is to derive the modification, caused by the finite temperature and density corrections, of the Coulomb potential. It is a well known fact that zero temperature quantum corrections to the classical Coulomb law at small distances change the form of the point charged-particle potential. (The modification of the Coulomb potential at small distances in the Johnson–Baker–Willey model of QED has been studied recently by Manoukian (1982).) These corrections were first calculated in 1935 (Serber 1935, Uehling 1935) and have led to the concept of the renormalisation group equations in quantum field theory (Gell-Mann and Low 1954). The renormalisation group equations play the fundamental role in the asymptotically free theories (Gross and Wilczek 1973a, b, Politzer 1973). At large distances, however, the zero temperature corrections to the Coulomb law vanish.

In this section finite temperature and density effects are studied and it is shown that heat corrections change the Coulomb law at large distances and that the modified potential takes the form

$$\mathfrak{A}_0^c(r) = (e/r)\exp[-a(T, \mu)r] \tag{24}$$

where the function $a(T, \mu)$ vanishes at $T = 0$.

The effective potential $\mathfrak{A}_\mu$ in QED at the vanishing external current is defined to be (Białynicki-Birula and Białynicka-Birula 1975)

$$\mathfrak{A}_\mu[z|\mathscr{A}] = V[\mathscr{A}]^{-1} \exp\left((1/2i)\int dx\, dx' \frac{\delta}{\delta\mathscr{A}_\nu(x)} D^F_{\nu\rho}(x-x') \frac{\delta}{\delta\mathscr{A}_\rho(x')}\right)\mathscr{A}_\mu(z)C[\mathscr{A}],$$

where $\mathscr{A}_\mu$ and $D^F_{\nu\rho}$ are the external electromagnetic potential and the photon propagator, respectively. Moreover,

$$V[\mathscr{A}] = \exp\left(\frac{1}{2i}\int dx\, dx' \frac{\delta}{\delta\mathscr{A}_\nu(x)} D^F_{\nu\rho}(x-x') \frac{\delta}{\delta\mathscr{A}_\rho(x')}\right)C[\mathscr{A}]$$

† This is the Feynman propagator of the electron driven by the monochromatic wave of the amplitude $\boldsymbol{E}$ and the frequency $\omega$ provided that $|\boldsymbol{E}| \gg \omega F_0/m$.

and

$$C[\mathscr{A}] = \exp\left(\mathrm{i} \int \mathrm{d}x\, \mathscr{L}^{(1)}_{\mathrm{eff}}[x|\mathscr{A}]\right)$$

$$= \exp\left(\mathrm{i} \int \mathrm{d}x\, \mathrm{Tr}[\ln K_{\mathrm{F}}[x, x|\mathscr{A}] - \ln K_{\mathrm{F}}[x, x|0]]\right)$$

where $K_{\mathrm{F}}$ and $\mathscr{L}^{(1)}_{\mathrm{eff}}[x|\mathscr{A}]$ are the Feynman propagator of the electron in the external EM field and the effective Lagrangian, respectively. At the vanishing external current and in the linear approximation (i.e., terms that depend nonlinearly both on $\mathscr{A}$ and $\mathfrak{A}$ are neglected) it can be shown that

$$\mathscr{A}_\mu(z) = \mathfrak{A}_\mu(z) - \int \mathrm{d}z_1\, \mathrm{d}z_2\, D^F_{\mu\lambda}(z - z_1)\pi^{\lambda\nu}(z_1, z_2)\mathfrak{A}_\nu(z_2) \tag{25}$$

where $\pi^{\lambda\nu}$ is the polarisation tensor at the vanishing external EM field and current.

Further I will be concerned with the static potential with the non-vanishing time component, i.e.

$$\mathscr{A}_\mu(x) = g_{\mu 0}\mathscr{A}_0(\boldsymbol{r}).$$

In this case (25) can be rewritten in the following form:

$$\tilde{\mathfrak{A}}_0(\boldsymbol{k}) = \tilde{\mathscr{A}}_0(\boldsymbol{k}) + \int \mathrm{d}k_0\, \delta(k_0)\tilde{\mathfrak{A}}_0(\boldsymbol{k})\tilde{D}^F_{00}(k)\tilde{\pi}^{00}(k) \tag{26}$$

where

$$\tilde{D}^F_{00}(k) = (k^2 + \mathrm{i}\varepsilon)^{-1}.$$

$\tilde{\pi}^{00}(k)$ in second order of perturbation theory has the form

$$\tilde{\pi}^{00}(k) = \mathrm{i}\, e^2 \int \frac{\mathrm{d}p}{(2\pi)^4} \mathrm{Tr}\, \gamma^0(\gamma p + \gamma k - m + \mathrm{i}\varepsilon)^{-1}\gamma^0(\gamma p - m + \mathrm{i}\varepsilon)^{-1}. \tag{27}$$

For the Coulomb potential

$$\tilde{\mathscr{A}}^{\mathrm{c}}_0(\boldsymbol{k}) = \frac{e}{\boldsymbol{k}^2},$$

hence, at large distances ($\boldsymbol{k}^2 \ll m^2$), we obtain

$$\tilde{\mathfrak{A}}^{\mathrm{c}}_0(\boldsymbol{k}) = e/[\boldsymbol{k}^2 + \tilde{\pi}^{00}(0)],$$

or in the configuration space,

$$\mathfrak{A}^{\mathrm{c}}_0(r) = (e/4\pi r) \exp[-(\tilde{\pi}^{00}(0))^{1/2} r] \tag{28}$$

where at low temperature,

$$\tilde{\pi}^{00}(0) = -4(2\pi)^{-1/2}\alpha m^2 (T/m)^{1/2} F(\tfrac{1}{2}) + \mathrm{O}(\alpha^2). \tag{29}$$

At the vanishing chemical potential and low temperature, with respect to the electron mass, we obtain the following Yukawa-type effective potential†

$$\mathfrak{A}^{\mathrm{c}}_0(r) = (e/r) \exp\{-2m[(2\alpha^2 T/m\pi)\, \mathrm{e}^{-2m/T}]^{1/4} r\}. \tag{30}$$

† The mass generation in finite temperature QED and at $\mu = 0$ has been discussed by Babu Joseph (1982).

## 6. Comments on the anomalous magnetic moment

This section is devoted to the calculations of the finite temperature corrections to the electron magnetic moment, that have been performed by Peressutti and Skagerstam (1982). In the textbooks (see e.g. Białynicki-Birula and Białynicka-Birula 1975) one can find that the energy correction due to the interaction of the electron with the weak external magnetic field is equal to

$$\Delta E = -\sum_{r,s} \int \mathrm{d}\Gamma(m) f^*(\boldsymbol{p}, r) w^+(\boldsymbol{p}, r)\left(\frac{e}{2m} F_1(0)(\mathrm{i}\boldsymbol{\nabla}_p \times \boldsymbol{p}) + \frac{e}{2m}(F_1(0)+F_2(0))\boldsymbol{\sigma}\right) w(\boldsymbol{p}, s) f(\boldsymbol{p}, s) \cdot \boldsymbol{H}. \tag{31}$$

The first term describes the potential energy of the magnetic moment associated with the orbital moment of the electron in a magnetic field $\boldsymbol{H}$ and the second term describes the potential energy of the interaction of the anomalous magnetic moment of the electron. So we see that the operator of the magnetic moment $\boldsymbol{\mu}$ is given by the formula

$$\boldsymbol{\mu} = (e/2m)(F_1(0)+F_2(0))\boldsymbol{\sigma} \tag{32}$$

where $F_1$ and $F_2$ are the form factors of the vertex function. At zero temperature $F_1(0) = 1$, and the electron $g$ factor is defined to be

$$g = 2(1+F_2(0)). \tag{33}$$

This formula has been used by Peressutti and Skagerstam (1982) in their calculation of the temperature corrections to the electron $g$ factor. However, it follows from (31) that at finite temperature one must also take into account the contribution coming from the $F_1$ form-factor, i.e., at finite temperature and/or density the $g$ factor is defined ambiguously. If in the experiment the energy shift induced by the anomalous magnetic moment is measured, then the $g$ factor is defined to be

$$g^{\mathrm{I}}_{T,\mu} = 2(F_1(0)+F_2(0)). \tag{34}$$

On the other hand if the ratio of the energy shifts induced by the orbital moment and the anomalous magnetic moment, respectively, is measured then the $g$ factor is defined to be

$$g^{\mathrm{II}}_{T,\mu} = 2(1+F_2(0)/F_1(0)). \tag{35}$$

Since $F_1(0) = 1 + \mathrm{O}(\alpha)$, it follows from (35) that Peressutti and Skagerstam have calculated the $g^{\mathrm{II}}_{T,\mu}$ factor in second order of perturbation theory.

## Appendix

The Feynman propagator of the electron in the constant EM field has been obtained in many papers. The reader who is interested in this subject is referred to Kamiński (1981c) where one can find the original references. In this appendix only the final result is given

$$\tilde{K}_{\mathrm{F}}[p|\boldsymbol{E}, \boldsymbol{B}] = \mathrm{i}\int_0^\infty \mathrm{d}s\,(m + p^\mu A_\mu) \exp(-\mathrm{i}m^2 s - \tfrac{1}{2}\mathrm{i}es\sigma^{\mu\nu}f_{\mu\nu} + \mathrm{i}B_{\mu\nu}p^\mu p^\nu + C) \tag{A1}$$

where the four-dimensional matrices $A$, $B$, $C$ are equal to

$$A = \gamma + e\gamma f B, \qquad B = (ef)^{-1}\tanh(efs), \qquad C = -\tfrac{1}{2}\,\mathrm{Tr}\,\ln\cosh(efs).$$

Moreover, $B = (B^{\mu}_{\nu})$ and in the Lorentz frame where $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel to each other,

$$f = (f^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & 0 & 0 & E \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix}.$$

The unitary matrix $\mathbf{U}$ ($\mathbf{U}^{+}\mathbf{U} = \mathbf{U}\mathbf{U}^{+} = \mathbf{I}$) that diagonalises $f$ and the diagonalised matrix $f_{\mathrm{D}} = \mathbf{U}^{+} f \mathbf{U}$ are equal to

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{f}_{\mathrm{D}} = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & \mathrm{i}B & 0 & 0 \\ 0 & 0 & -\mathrm{i}B & 0 \\ 0 & 0 & 0 & -E \end{pmatrix}.$$

The matrix $\mathbf{U}$ can be used to define an arbitrary function of $f$, i.e.,

$$\varphi(f) = \mathbf{U}\varphi(f_{\mathrm{D}})\mathbf{U}^{+} = \begin{pmatrix} \varphi_{+}(E) & 0 & 0 & \varphi_{-}(E) \\ 0 & \varphi_{+}(\mathrm{i}B) & -\mathrm{i}\varphi_{-}(\mathrm{i}B) & 0 \\ 0 & \mathrm{i}\varphi_{-}(\mathrm{i}B) & \varphi_{+}(\mathrm{i}B) & 0 \\ \varphi_{-}(E) & 0 & 0 & \varphi_{+}(E) \end{pmatrix}$$

where $\varphi_{\pm}(x) = \frac{1}{2}(\varphi(x) \pm \varphi(-x))$. With the help of this result one obtains

$$\mathrm{i}B_{\mu\nu}p^{\mu}p^{\nu} = \mathrm{i}s\varepsilon^{-1}(p_0^2 - p_3^2)\tanh\varepsilon - \mathrm{i}s\eta^{-1}(p_1^2 + p_2^2)\tan\eta,$$

$$e^{C} = (\cosh\varepsilon\,\cos\eta)^{-1},$$

$$(\gamma + \gamma\tanh(efs))p$$
$$= (p_0 - p_3\tanh\varepsilon)\gamma^{0} + (p_1 + p_2\tan\eta)\gamma^{1} + (p_2 - p_1\tan\eta)\gamma^{2} + (p_3 - p_0\tanh\varepsilon)\gamma^{3},$$

where $\varepsilon = eEs$ and $\eta = eBs$. Taking advantage of properties of the Dirac matrices one finds (the notation is taken from (Bjorken and Drell 1965))

$$\exp(-\tfrac{1}{2}\mathrm{i}es\sigma^{\mu\nu}f_{\mu\nu}) = X + \mathrm{i}Z\sigma^{12} + \mathrm{i}Y\sigma^{30} + \mathrm{i}\gamma_5 V$$

where

$$X = \cos\eta\,\cosh\varepsilon, \qquad V = \sin\eta\,\sinh\varepsilon,$$
$$Y = \cos\eta\,\sinh\varepsilon, \qquad Z = \sin\eta\,\cosh\varepsilon.$$

Assembling these results one can write the Feynman propagator in the following form:

$$\tilde{K}_{\mathrm{F}}[p|\boldsymbol{E},\boldsymbol{B}] = \mathrm{i}\int_0^{\infty} \mathrm{d}s\,\exp(-\mathrm{i}m^2 s + \mathrm{i}s\varepsilon^{-1}\tanh\varepsilon\,(p_0^2 - p_3^2) - \mathrm{i}s\eta^{-1}\tan\eta\,(p_1^2 + p_2^2))(mX + v_{\mu}\gamma^{\mu} + \mathrm{i}t_{\mu\nu}\sigma^{\mu\nu} + \mathrm{i}a_{\mu}\gamma^{\mu}\gamma_5 + \mathrm{i}mV\gamma_5), \quad \text{(A2)}$$

where

$$t_{\mu\nu} = \tfrac{1}{2}m[Z(g_{\mu 1}g_{\nu 2} - g_{\mu 2}g_{\nu 1}) + Y(g_{\mu 3}g_{\nu 0} - g_{\mu 0}g_{\nu 3})],$$
$$v_0 = p_0(X - Y \tanh \varepsilon) - p_3(X \tanh \varepsilon - Y),$$
$$v_1 = p_1(X + Z \tan \eta) + p_2(X \tan \eta - Z),$$
$$v_2 = -p_1(X \tan \eta - Z) + p_2(X + Z \tan \eta),$$
$$v_3 = -p_0(X \tanh \varepsilon - Y) + p_3(X - Y \tanh \varepsilon),$$
$$a_0 = p_0(V - Z \tanh \varepsilon) - p_3(V \tanh \varepsilon - Z),$$
$$a_1 = p_1(V - Y \tan \eta) + p_2(V \tan \eta + Y),$$
$$a_2 = -p_1(V \tan \eta + Y) + p_2(V - Y \tan \eta),$$
$$a_3 = -p_0(V \tanh \varepsilon - Z) + p_3(V - Z \tanh \varepsilon).$$

Taking advantage of (A2) one can calculate the electric current. Using the finite temperature and density methods one finds:

$$j^\mu = -eT \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr} \gamma^\mu \tilde{\boldsymbol{K}}_F[\mu + 2\pi i T(n+\tfrac{1}{2}), \boldsymbol{p}|\boldsymbol{E}, \boldsymbol{B}]$$
$$= -4e\delta_0^\mu T \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{ds}{(4\pi i s)^{3/2}} e^{-im^2 s}(\varepsilon \coth \varepsilon)^{1/2} \eta \cot \eta$$
$$\times [\mu + 2\pi i T(n+\tfrac{1}{2})] \exp\{is\varepsilon^{-1} \tanh \varepsilon [\mu + 2\pi i T(n+\tfrac{1}{2})]^2\}. \qquad \text{(A3)}$$

Comparing (A3) with (3) one arrives at

$$j^\mu = \delta_0^\mu e(\partial/\partial\mu)\mathscr{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}, T, \mu). \qquad \text{(A4)}$$

Since the charge density is proportional to the number density, (A4) is followed by (7).

## References


Abramowitz M A and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Adler S L 1971 *Ann. Phys., NY* **67** 599–647
—— 1972 *Phys. Rev.* D **5** 3021–43
Babu Joseph K 1982 *Phys. Lett.* **115B** 252–6
Bateman H 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
Bauhoff W and Dittrich W 1981 *Lett. Nuovo Cimento* **30** 399–402
Bechler A 1981 *Ann. Phys., NY* **135** 19–57
Białynicka-Birula Z and Białynicki-Birula I 1970 *Phys. Rev.* D **2** 2341–5
Białynicki-Birula I and Białynicka-Birula Z 1975 *Quantum Electrodynamics* (Oxford: Pergamon)
Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
Dittrich W 1976 *J. Phys. A: Math. Gen.* **9** 1171–9
—— 1977 *J. Phys. A: Math. Gen.* **10** 833–41
—— 1979a *Phys. Lett.* **83B** 67–70
—— 1979b *Phys. Rev.* D **19** 2385–90
Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320–41
Gell-Mann M and Low F E 1954 *Phys. Rev.* **95** 1300–12
Gross D J and Wilczek F 1973a *Phys. Rev. Lett.* **30** 1343–6
—— 1973b *Phys. Rev.* D **8** 3630–52
Hagiwara T 1981 *Nucl. Phys.* B **189** 135–56

Harrington B J and Yildiz A 1974 *Phys. Rev. Lett.* **33** 324–7
Kamiński J Z 1981a *Phys. Lett.* **102B** 172–4
—— 1981b *Lett. Nuovo Cimento* **31** 582–4
—— 1981c *Acta Phys. Austriaca* **53** 287–304
—— 1982a *Acta Phys. Austriaca* **54** 1–5
—— 1982b *Acta Phys. Pol.* B **13** 379–82
Kirzhnits D A and Linde A D 1972 *Phys. Lett.* **42B** 471–4
Lautrup B E, Peterman A and de Rafael E 1972 *Phys. Rep.* **3C** 193–260
Manoukian E B 1982 *Phys. Rev.* D **25** 3420–2
Mielniczuk W J 1982 *J. Phys. A: Math. Gen.* **15** 2905–10
Peressutti G and Skagerstam B S 1982 *Phys. Lett.* **110B** 406–11
Politzer H D 1973 *Phys. Rev. Lett.* **30** 1346–9
Ritus V I 1978 *Multiphoton Processes* ed J H Eberly and P Lambropoulos (New York: Wiley)
Schwartz L 1966 *Mathematics for the Physical Sciences* (Massachussetts: Addison-Wesley)
Schwinger J 1951 *Phys. Rev.* **82** 664–79
Serber R 1935 *Phys. Rev.* **48** 49–54
Uehling A E 1935 *Phys. Rev.* **48** 55–63
Weinberg S 1974 *Phys. Rev.* D **9** 3357–78